Trinity College

MATHEMATICS SPECIALIST UNIT 1,2
Section One:
Calculator-free

SOLUTIONS

Student number: In figures   ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐

In words   _____

Your name   _____

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	101	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section One: Calculator-free   **35% (53 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1   **(5 marks)**

Let the displacement vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be $(11, -4)$, $(5, 14)$ and $(8, m)$ respectively, where m is a constant.

(a) Determine the vector $3\mathbf{a} + 2\mathbf{b}$.   (2 marks)

Solution
$3\mathbf{a} + 2\mathbf{b} = 3(11, -4) + 2(6, 14)$
$= (33, -12) + (12, 28)$
$= (43, 16)$
Specific behaviours
✓ multiplies by scalar
✓ correct vector

(b) Given that $(\mathbf{a} + \mathbf{b} + k\mathbf{c}) = 0$, determine the values of k and m.   (3 marks)

Solution
$\begin{pmatrix} 11 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 \\ 14 \end{pmatrix} + k \begin{pmatrix} 8 \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
From $\mathbf{i}$-coeff: $11 + 5 + 8k = 0 \Rightarrow k = -2$
From $\mathbf{j}$-coeff: $-4 + 14 - 2m = 0 \Rightarrow m = 5$
Specific behaviours
✓ vector equation
✓ value of k
✓ value of m

Question 2 **(6 marks)**

Relative to the origin O, points A and B have position vectors $-3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively.

(a) Determine the unit vector $\hat{\mathbf{c}}$, where $\mathbf{c} = \overrightarrow{AB}$. (3 marks)

Solution
$$\mathbf{c} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$
$$\|\mathbf{c}\| = \sqrt{16 + 4} = 2\sqrt{5}$$
$$\hat{\mathbf{c}} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{\sqrt{5}}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
Specific behaviours
✓ vector $\mathbf{c}$
✓ magnitude
✓ unit vector, simplified

(b) Vector $\mathbf{d}$ has magnitude $3\sqrt{5}$, is parallel to $\mathbf{c}$ and in the opposite direction. Determine $\mathbf{d}$.

(3 marks)

Solution
$$\mathbf{d} = 3\sqrt{5} \times (-1) \times \frac{\sqrt{5}}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$ $$= \begin{pmatrix} -6 \\ 3 \end{pmatrix}$$
Specific behaviours
✓ reverses $\mathbf{c}$
✓ correct vector

Question 3 **(8 marks)**

Consider the following statement about a simple (no edges that cross) polygon:

If it has an interior angle sum of 360°, then it is a square.

(a) Use a counter-example to explain why the statement is false. (2 marks)

Solution
A trapezium has an interior angle sum of 360° but is not a square.
Specific behaviours
✓ names or draws any quadrilateral that is not a square
✓ uses angle sum and fact that shape is not a square

(b) Write the converse statement and state whether it is always, sometimes or never true. (2 marks)

Solution
If it is a square, then it has an interior angle sum of 360°.
The converse statement is **always** true.
Specific behaviours
✓ writes converse
✓ states always true

(c) Write the inverse statement and state whether it is always, sometimes or never true. (2 marks)

Solution
If it does not have an interior angle sum of 360°, then it is not a square.
The inverse statement is **always** true.
Specific behaviours
✓ writes inverse
✓ states always true

(d) Write the contrapositive statement and state whether it is always, sometimes or never true. (2 marks)

Solution
If it is not a square, then it does not have an interior angle sum of 360°.
The contrapositive statement is **sometimes** true. *(eg true for triangle, false for any quadrilateral)*
Specific behaviours
✓ writes contrapositive
✓ states sometimes true

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Question 4   **(6 marks)**

(a)   Determine the value of the constant n, given that the vectors $12\mathbf{i} + n\mathbf{j}$ and $5\mathbf{i} - 8\mathbf{j}$ are perpendicular.   (2 marks)

Solution
$60 - 8n = 0$
$n = \dfrac{15}{2} = 7.5$
Specific behaviours
✓ equates scalar product to 0
✓ solves for n

(b)   The vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $|\mathbf{a}| = 18, |\mathbf{b}| = 12$ and $\mathbf{a} \cdot \mathbf{b} = -3$. Evaluate

(i) $-2\mathbf{a} \cdot 3\mathbf{b}$.   (1 mark)

Solution
$-3 \times -2 \times 3 = 18$
Specific behaviours
✓ correct value

(ii) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a})$.   (3 marks)

Solution
$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}$
$= |\mathbf{b}|^2 - |\mathbf{a}|^2$
$= 12^2 - 18^2$
$= (12 + 18)(12 - 18)$
$= 30 \times -6 = -180$
Specific behaviours
✓ expands
✓ simplifies to difference of squares
✓ correct value

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Question 5 **(6 marks)**

A drone leaves point P and travels 74 m on bearing of 290° to Q, then 40 m on bearing 020° to R and finally 114 m on bearing 110° to S.

(a) Sketch a neat diagram to show the path of the drone. (2 marks)

Solution
Specific behaviours
✓ first leg, labelled
✓ other legs and labels

(b) The drone is to return directly from S to P. Determine the distance it must fly and on what bearing. (4 marks)

Solution	
	$PS = 40\sqrt{2}$ m $\angle SPT = 45°$ Bearing is $180 + 45 + 20 = 245°$
Specific behaviours	
✓ simplified sketch of return leg [or seen in (a)]	
✓ correct distance	
✓ indicates angle of 45° in triangle	
✓ correct bearing	

See next page

Question 6   **(7 marks)**

(a)   In the diagram below, the vertices of triangle ABC lie on a circle with centre O. Given that
       $\angle ABC = 63°$, determine the values of $\angle AOC$ and $\angle OAC$.   (2 marks)
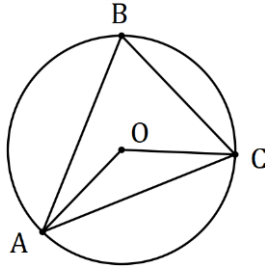
Solution
$\angle AOC = 2 \times 63 = 126°$
$\angle OAC = \dfrac{180 - 126}{2} = 27°$

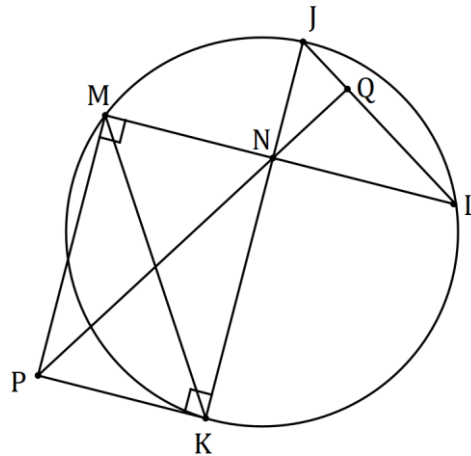| Specific behaviours |
|---|
| ✓ first angle |
| ✓ second angle |

(b)   Prove, assuming only basic axioms and properties of triangles, that the size of the angle
       at the centre subtended by an arc of a circle is twice the size of the angle at the
       circumference subtended by the same arc.   (5 marks)

Solution

Required to prove that $a = 2b$

Let $b = \angle OQP + \angle OQR$

But $\angle OQP = \angle OPQ$ and $\angle OQR = \angle ORQ$ (isosceles triangles)

And so $\angle QOP = 180 - 2\angle OQP$ and $\angle QOR = 180 - 2\angle OQR$

$$\text{At } O, a = 360 - \angle QOP - \angle QOR$$
$$a = 360 - (180 - 2\angle OQP) - (180 - 2\angle OQR)$$
$$a = 2(\angle OQP + \angle OQR)$$
$$a = 2b, \text{ as required.}$$

Specific behaviours
✓ labelled diagram(s) illustrating RTP
✓ uses isosceles triangles
✓ expressions for angles at O
✓ equation using angle sum at a point
✓ substitutes and simplifies

Question 7 **(7 marks)**

In the diagram below, two chords of a circle, JK and LM, intersect at N. PM is perpendicular to LM at M and PK is perpendicular to JK at K. The line PN intersects chord JL at Q.

(a)   Explain why $PMNK$ is a cyclic quadrilateral.   (1 mark)

Solution
Sum of opposite angles $\angle PMN + \angle PKN = 180°$.
Specific behaviours
✓ explanation using opposite angles and their sum

(b)   Prove that $\angle MPN = \angle JLM$.   (3 marks)

Solution
$\angle MPN = \angle MKN$ (common arc MN)
$\angle JLM = \angle JKM$ (common arc JM)
Hence $\angle MPN = \angle JLM$ (since $\angle MKN$ and $\angle JKM$ are same angle)
Specific behaviours
✓ uses circle from (a)
✓ uses circle shown
✓ reasoning

(c)   Prove that PQ is perpendicular to JL.   (3 marks)

Solution
$\angle MNP = \angle QNL$ (Vert Opp)
Hence $\triangle MNP \sim \triangle QNL$ (AA)
$\angle PMN = \angle LQN = 90° \Rightarrow \perp$
Specific behaviours
✓ uses vertically opposite angles
✓ uses AA for similarity
✓ deduces perpendicular

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Question 8   **(9 marks)**

(a)   Evaluate $^{25}P_{19} \div {}^{23}P_{20}$.   (3 marks)

Solution
$\dfrac{25!}{6!} \times \dfrac{3!}{23!} = \dfrac{25 \times 24}{6 \times 5 \times 4} = 5$
Specific behaviours
✓ expresses using factorials
✓ eliminates factorials
✓ evaluates

(b)   Express $8! + 7! + 6!$ in the form $a^2 b!$, where a and b are positive integers.   (3 marks)

Solution
$8! + 7! + 6! = (8 \times 7 + 7 + 1) \times 6!$
$= 64 \times 6!$
$= 8^2 \times 6!$
Specific behaviours
✓ factors out lowest factorial
✓ simplifies
✓ writes in required form

(c)   Show that for n an $integer$, $n \geq 1$, the sum $(n + 2)! + (n + 1)! + n!$ can always be expressed in the form $a^2 b!$ where a and b are positive integers.   (3 marks)

Solution
$(n + 2)! + (n + 1)! + n! = \big((n + 2)(n + 1) + (n + 1) + 1\big)n!$
$= (n^2 + 4n + 4)n!$
$= (n + 2)^2 n!$
$(a = n + 2 \text{ and } b = n)$
Specific behaviours
✓ uses $n!$ as one factor
✓ clearly shows composition of second factor
✓ simplifies second factor and writes as required

End of questions

Supplementary page

Question number: _____